UDC 539.3

ASYMPTOTIC INTEGRATION OF STATIC EQUATIONS OF THE THEORY OF ELASTICITY IN CARTESIAN COORDINATES WITH AUTOMATED SEARCH OF INTEGRATION PARAMETERS

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The formal method of search of asymptotic integration parameters which makes possible a comprehensive classification of all possible variants of simplified equations on the basis of the minimal simplification criterion is considered on examples of static equations of the theory of elasticity for orthotropic and isotropic media. The method which is suitable for use on computers provides the possibility of operating with a large number of parameters. In the case of three-dimensional equations of the theory of elasticity for an isotropic medium it is possible to obtain by this method all known (presented earlier in [1]) variants, as well as a number of other.

One of the effective modern methods of constructing two-dimensional equations of the theory of plates and shells on the basis of three-dimensional equations of the theory of elasticity is the method of asymptotic integration developed by Gol'denveizer [1]. A similar method for deriving simplified equations by rejecting unimportant terms can be applied in many other problems. However, its wider use is hampered by the difficulty encountered in the search of asymptotic integration parameters. Hence the formal search of these parameters, valid in the case of minimal preliminary information about the solution, is of interest. The feasibility of this was shown in [2] on the example of the dynamical problem of the thin plate problem in the theory of elasticity.

1. To illustrate the problem we begin with the simplest example of the differential equation

$$B_1 \frac{\partial^2 u}{\partial x^2} + G \frac{\partial^2 u}{\partial y^2} + (B_1\mu_2 + G) \frac{\partial^2 v}{\partial x \partial y} = 0 \tag{1.1}$$
$$B_2 \frac{\partial^2 v}{\partial y^2} + G \frac{\partial^2 v}{\partial x^2} + (B_2\mu_1 + G) \frac{\partial^2 u}{\partial x \partial y} = 0$$

of the plane problem of the theory of elasticity for an orthotropic medium [3]. In this equation u and v are components of the displacement vector, B_1 and B_2 are tension and compression stiffnesses, and G is the shear stiffness. We assume that relations $B_1 > B_2 \sim G$ are satisfied. We introduce the small parameter $\varepsilon = B_2 / B_1$ and carry out the transformations

$$x^* = \varepsilon^\alpha x, \quad y^* = y; \quad u^* = \varepsilon^\beta u, \quad v^* = v \tag{1.2}$$

so as to have the relations

$$\partial / \partial x^* \sim \partial / \partial y^*, \quad u^* \sim v^* \tag{1.3}$$

satisfied.

Substitution of (1.2) into (1.1) yields the equations

$$\begin{gathered}\varepsilon^{2\alpha-\beta}\frac{\partial^2 u^*}{\partial x^{*2}} + \varepsilon^{1-\beta}\frac{G}{B_2}\frac{\partial^2 u^*}{\partial y^{*2}} + \varepsilon^{1+\alpha}\left(\mu_1 + \frac{G}{B_2}\right)\frac{\partial^2 v^*}{\partial x^*\partial y^*} = 0 \\ \frac{\partial^2 v^*}{\partial y^{*2}} + \varepsilon^{2\alpha}\frac{G}{B_2}\frac{\partial^2 v^*}{\partial x^{*2}} + \varepsilon^{\alpha-\beta}\left(\mu_1 + \frac{G}{B_2}\right)\frac{\partial^2 u^*}{\partial x^*\partial y^*} = 0\end{gathered} \tag{1.4}$$

where in conformity with (1.3) the contribution of each term is evaluated by the power of ε in which it appears in that term. We represent the unknown functions in the form of asymptotic series

$$u^* = \sum_{i=1}^{\infty} u_i^* \varepsilon^{i-1}, \quad v^* = \sum_{i=1}^{\infty} v_i^* \varepsilon^{i-1} \tag{1.5}$$

As $\varepsilon \to 0$ we have $u^* \to u_1^*$, $v^* \to v_1^*$. Equations for u_1^* and v_1^* are obtained by substituting (1.5) into (1.4) and retaining terms that contain ε of the lowest power, since these terms remain when $\varepsilon \to 0$. Selection of these terms substantially depends on parameters α and β. Similar parameters are usually chosen based on intuitive considerations about the properties of the sought solution of the input problems. For instance, assuming that the rate of change of unknown functions is higher along the x-axis than along the y-axis, we select $\alpha < 0$ and in the opposite case $\alpha > 0$, and when these rates are equal we take $\alpha = 0$. Parameter β defines the comparative magnitudes of displacements u and v. Values $\beta < 0$ correspond to the preponderance of v over u, $\beta > 0$ to that of u over v, and $\beta = 0$ to displacements of the same order.

Such method requires considerable preliminary information about the sought solution, and the difficulties of its application increase with the number of parameters. The problem is furthermore complicated by the fact that in various cases the solution properties and, consequently, parameter values may change, resulting in different simplified equations for the same input system. The determination of all such variants is highly desirable, since the various simplified equations complement each other, and in their totality define (approximately) the range of problems specified by the input system of equations.

Let us formulate the task of investigating all possible values of α and β. The exponents of ε in all terms of the first and second of Eqs. (1.4) are

$$\begin{matrix} 2\alpha - \beta, & 1 - \beta, & 1 + \alpha \\ 0, & 2\alpha, & \alpha - \beta \end{matrix} \tag{1.6}$$

Let us consider the $\alpha\beta$-plane and construct in it separate zones containing the lowest values of exponents in each equation (see Fig. 1). Exponent $2\alpha - \beta$ is the smallest in the first row of (1.6) for α and β selected from zone 1.1. Exponents $1 - \beta$ and $1 + \alpha$ are the smallest in zones (1.2) and (1.3), respectively. Similarly, the zero exponent in the second row is the smallest in zone 2.1, and exponents 2α and $\alpha - \beta$ are the smallest in zones 2.2 and 2.3, respectively.

Let us, first, consider the parameters for which the representing point in the $\alpha\beta$-plane is simultaneously inside some zone that corresponds to the first equation and inside one of the zones that correspond to the second equation. We than have in (1.6) one minimal exponent in the first row and one in the second, and the equations for u_1^* and v_1^* have only one term each, which shows that the input equation is simplified to the maximum possible extent. Such extreme simplification in the first stage is undesirable. Selecting the representing point on the common boundary of two zones corresponding to one of the equations diminishes the approximation [accuracy]. The first approximation equation then contains two terms. The minimal simplification obtains when the representing point is at the intersection of two boundaries. There are four such points, as shown in Fig. 1. Of the greatest interest are points A and B at which the boundaries separating zones of different equations intersect.

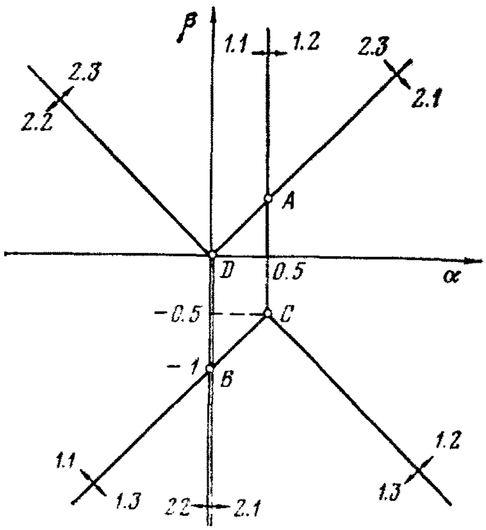

Fig. 1

Point A ($\alpha = 0.5$, $\beta = 0.5$). The first approximation equations in terms of input variables are of the form

$$B_1 \frac{\partial^2 u_1}{\partial x^2} + G \frac{\partial^2 u_1}{\partial y^2} = 0, \qquad (1.7)$$

$$B_2 \frac{\partial^2 v_1}{\partial y^2} + (B_2\mu_1 + G) \frac{\partial^2 u_1}{\partial x\, \partial y} = 0$$

Point B ($\alpha = 0$, $\beta = -1$). The first approximation equations are of the form

$$B_2 \frac{\partial^2 v_1}{\partial y^2} + G \frac{\partial^2 v_1}{\partial x^2} = 0, \quad B_1 \frac{\partial^2 u_1}{\partial x^2} + (B_1\mu_2 + G) \frac{\partial^2 v_1}{\partial x\, \partial y} = 0 \qquad (1.8)$$

These equations were used in [4] and other investigations for solving contact problems of the theory of elasticity, as well as for defining the slowly changing with respect to x basic state (1.7) and (1.8) for defining the boundary layer, when considering regions with boundaries $x = \text{const}$.

Of lesser interest are points C and D which define α and β for which all terms are retained in one of the equations and only one in the other.

Let us consider the conclusions that can be drawn from the above simplest example.

The criterion of selection of specific asymptotic integration parameters from an infinite number of variants has been reasonably formulated. At the first stage of investigation such parameters are selected so as to obtain a minimal simplication of the input system of equations. If only two parameters are involved, the search for these can be carried out graphically, as described above. A larger number of parameters necessitates the use of analytic methods for their determination. We shall show the essence of such methods using the same two-dimensional example. Let us, first, write the equations for the boundaries that separate zones shown in Fig. 1, equating pairwise the exponents in the first or second rows in (1.6) and stipulating that the two equal exponents must not exceed the third in the same row. As a result, we obtain six equations

$$\begin{array}{llll} \alpha = 0.5, & \beta \geqslant -0.5; & \alpha = 0, & \beta \leqslant 0 \\ \alpha - \beta = 1, & \beta \leqslant -0.5; & \alpha - \beta = 0, & \beta \geqslant 0 \\ \alpha + \beta = 0, & \beta \leqslant -0.5; & \alpha + \beta = 0, & \beta \geqslant 0 \end{array} \tag{1.9}$$

accompanied by inequalities three of which correspond in (1.6) to the first row and three to the second.

To determine the boundary intersection points we solve Eqs. (1.9) in pairs using all possible combinations. In this case there are fifteen of these, but part of the equation pairs are incompatible because of the accompanying inequalities. There are also cases when the same α and β are obtained from solutions of different equation pairs (for instance, solutions that correspond to points C and D occur three times). As a result, we are left with only four different pairs of α and β which correspond to point A, B, C, and D.

The described analytic method can be readily extended to any arbitrary number of parameters. An example of this is presented below.

2. The absence of some intrinsic small parameter in the input differential equations does not prevent the application of the asymptotic analysis methods in which the main part is played by transformations of the form (1.2) and (1.3). Such transformations can be effected by using as the basis any formally introduced parameter $\varepsilon < 1$.

As an example of the formal small parameter application, we shall consider the problem investigated in [1] of the asymptotic integration of static equations of the theory of elasticity, using Cartesian coordinates. These equations are of the form

$$\begin{gathered} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0, \quad E\,\frac{\partial u}{\partial x} = \sigma_x - \nu(\sigma_y + \sigma_z) \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0, \quad E\,\frac{\partial v}{\partial y} = \sigma_y - \nu(\sigma_x + \sigma_z) \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0, \quad E\,\frac{\partial w}{\partial z} = \sigma_z - \nu(\sigma_x + \sigma_y) \\ G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \tau_{xy}, \quad G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \tau_{xz}, \quad G\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \tau_{yz} \end{gathered} \tag{2.1}$$

and do not contain any intrinsic small parameter. In [1] which was aimed at the investigation of the stress-strain state of plates, the plate half-thickness was taken as the small parameter. However, in the general application of the results of asymptotic integration of Eqs. (2.1), without any specific aim, to the investigation of the stress-strain state of plates such dimensional parameter cannot be considered small, since it may assume any (including as large as desired) values depending on the scale of the problem. Hence we shall use the formally introduced dimensionless small parameter $\varepsilon < 1$.

We carry out the transformation

$$x^* = \varepsilon^{\alpha_1}x, \quad y^* = \varepsilon^{\alpha_2}y, \quad z^* = z; \quad u^* = \varepsilon^{\alpha_3}u, \quad v^* = \varepsilon^{\alpha_4}v, \tag{2.2}$$
$$w^* = \varepsilon^{\alpha_5}w, \quad \sigma_x^* = \varepsilon^{\alpha_6}\sigma_x, \quad \sigma_y^* = \varepsilon^{\alpha_7}\sigma_y, \quad \sigma_z^* = \varepsilon^{\alpha_8}\sigma_z$$
$$\tau_{xy}^* = \varepsilon^{\alpha_9}\tau_{xy}, \quad \tau_{xz}^* = \varepsilon^{\alpha_{10}}\tau_{xz}, \quad \tau_{yz}^* = \tau_{yz}$$

and assume that

$$a\frac{\partial}{\partial x^*} \sim a\frac{\partial}{\partial y^*} \sim a\frac{\partial}{\partial z^*} \sim 1 \tag{2.3}$$
$$\frac{u^*}{a} \sim \frac{v^*}{a} \sim \frac{w^*}{a} \sim \frac{\sigma_x^*}{E} \sim \frac{\sigma_y^*}{E} \sim \frac{\sigma_z^*}{E} \sim \frac{\tau_{xy}^*}{G} \sim \frac{\tau_{xz}^*}{G} \sim \frac{\tau_{yz}^*}{G}$$

where a is any constant of dimension length required for effecting the comparison of various dimensions in (2.3).

We seek the unknown functions in the form of series

$$u^* = \sum_{i=1}^{\infty} u_i^* \varepsilon^{i-1}, \ldots, \quad \tau_{yz}^* = \sum_{i=1}^{\infty} \tau_{yzi}^* \varepsilon^{i-1} \tag{2.4}$$

and follow the procedure used in the previous example for obtaining parameters $\alpha_1, \ldots, \alpha_{10}$ that determine the first approximation equations and the method of successive approximations. We substitute (2.2) into (2.1) and, taking into account (2.3), evaluate the contribution of each term of the transformed equations by the power of ε which it acquires as a multiplier. Omitting the presentation of transformed equations themselves, we adduce the exponents of ε in them tabulated in a form similar to (1.6)

$$\begin{array}{llll} \alpha_1-\alpha_6, & \alpha_2-\alpha_9, & -\alpha_{10}; & \alpha_1-\alpha_3, \quad -\alpha_6, \quad -\alpha_7, \quad -\alpha_8 \\ \alpha_1-\alpha_9, & \alpha_2-\alpha_7, & 0; & \alpha_2-\alpha_4, \quad -\alpha_7, \; -\alpha_6, \; -\alpha_8 \\ \alpha_1-\alpha_{10}, & \alpha_2, & -\alpha_8; & -\alpha_5, \quad -\alpha_8, \quad -\alpha_6, \quad -\alpha_7 \\ \alpha_2-\alpha_3, & \alpha_1-\alpha_4, & -\alpha_9; & -\alpha_3, \quad \alpha_1-\alpha_5, \quad -\alpha_{10}; \\ -\alpha_4, & \alpha_2-\alpha_5, & 0 & \end{array} \tag{2.5}$$

Using all possible combinations we equate pairwise the exponents corresponding to each input differential equation, and obtain for parameters $\alpha_1, \ldots, \alpha_{10}$ thirty six algebraic equations each of which we supplement by an inequality to conform with the condition that any two exponents chosen as equal should not exceed remaining exponents that correspond to one and the same differential equation. As an example we present the results related to the first row only of (2.5), i.e. to the first two of Eqs. (2.1)

$$\begin{array}{lll} \alpha_1-\alpha_2-\alpha_6+\alpha_9=0, & \alpha_2-\alpha_9+\alpha_{10}\leqslant 0 & \\ \alpha_1-\alpha_6+\alpha_{10}=0, & -\alpha_2+\alpha_9-\alpha_{10}\leqslant 0 & \\ \alpha_2-\alpha_9+\alpha_{10}=0, & -\alpha_1+\alpha_6-\alpha_{10}\leqslant 0 & \\ \cdots & \cdots & \\ \cdots & \cdots & \\ \cdots & \cdots & \\ \alpha_1-\alpha_3+\alpha_6=0, & -\alpha_6+\alpha_7\leqslant 0, & -\alpha_6+\alpha_8\leqslant 0 \\ \alpha_1-\alpha_3+\alpha_7=0, & \alpha_6-\alpha_7\leqslant 0, & -\alpha_7+\alpha_8\leqslant 0 \end{array} \tag{2.6}$$

$$\alpha_1-\alpha_3+\alpha_8=0, \qquad \alpha_6-\alpha_8\leqslant 0,\quad \alpha_7-\alpha_8\leqslant 0$$
$$\alpha_6-\alpha_7=0 \qquad -\alpha_1+\alpha_3-\alpha_7\leqslant 0,\quad -\alpha_7+\alpha_8\leqslant 0$$
$$\alpha_6-\alpha_8=0, \qquad -\alpha_1+\alpha_3-\alpha_8\leqslant 0,\quad \alpha_7-\alpha_8\leqslant 0$$
$$\alpha_7-\alpha_8=0. \qquad -\alpha_1+\alpha_3-\alpha_8\leqslant 0,\quad \alpha_6-\alpha_8\leqslant 0$$

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Owing to the formal introduction of the small parameter, all Eqs. (2.6) are homogeneous. Hence for the determination of the ten unknown $\alpha_1, \ldots, \alpha_{10}$ it is necessary, in conformity with the minimal simplification condition, to choose them from these equations in sets of nine instead of ten; the result of this operation is accurate to the common multiplier. The multiplier sign is selected so that the inequalities accompanying the equations are satisfied. The obtained solutions are equations of half-lines emanating from the coordinate origin of the ten-dimensional parameter space $\alpha_1, \ldots, \alpha_{10}$.

The problem of finding all variants of minimal simplification of the system of Eqs. (2.1) reduces to the determination of all different solutions of all possible sets of nine equations drawn out of thirty six equations (2.6). A similar problem was solved on a computer using the screening method. Provisions were made in the screening algorithm for separating the sets of nine that contained linearly dependent equations from those which yielded previously obtained solutions (since different sets of nine equations may yield the same solutions). The sets of nine which were incompatible with the accompanying inequalities were also separated.

As a result, 192 variants of solutions requiring further analysis were obtained. The first step consisted of elimination of parameter sets that led to first approximation equations with the number of unknown functions different from the number of equations. There were 108 such sets. Among the remaining 84 sets, 15 were rejected because they yielded nonzero values for some of the unknown functions not in the first approximation, while there were simultaneously sets the same [non-zero] values in the first approximation. The last remaining 69 sets were combined in 16 groups of three and six sets which yielded first approximation equations which converted into each other for any permutation of x, y, z.

Single representatives of each of these groups are tabulated below.

№	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}
1	−0.5	−0.5	0	0	−0.5	0.5	0.5	0.5	0.5	0
2	−0.5	−0.5	−1	−1	−0.5	−0.5	−0.5	−0.5	−0.5	0
3	0.5	0.5	0	0	0.5	0.5	0.5	0.5	0.5	0
4	0.5	0.5	1	0	0.5	0.5	0.5	−0.5	0.5	0
5	0.5	0.5	1	0	0.5	0.5	0.5	0.5	0.5	0
6	0.5	0.5	1	1	0.5	0.5	0.5	−0.5	0.5	0
7	0.5	0.5	1	1	0.5	0.5	0.5	0.5	0.5	0
8	0.5	0.5	1	1	1.5	0.5	0.5	−0.5	0.5	0

9	0.5	0.5	1	1	0.5	0.5	0.5	0.5	0.5	1
10	0.5	0.5	1	1	1.5	0.5	0.5	0.5	0.5	1
11	0.5	0.5	0	0	−0.5	−0.5	−0.5	−0.5	−0.5	0
12	0.5	0.5	0	0	0.5	−0.5	−0.5	−0.5	−0.5	0
13	0.5	0.5	1	1	1.5	0.5	0.5	0.5	0.5	0
14	0.5	1	0.5	0	1	0	0	−1	−0.5	−0.5
15	0.5	1	0.5	0	0	0	0	−1	−0.5	−0.5
16	0.5	1	0.5	0	1	0	0	0	0.5	−0.5

In any of these [groups] representatives yields first approximation equations that are invariant with respect to permutations of x and y, it generates two more sets, otherwise five sets are generated.

As previously indicated, parameters $\alpha_1, \ldots, \alpha_{10}$ are obtained with an accuracy to the common multiplier. The smallest absolute values of parameters, which enables us to split Eqs. (2.1) with respect to integral powers of ε after the substitution of (2.2) and (2.4) into (2.1), appear in the tabulation.

Let us examine the obtained results. Variants 1 and 2 lead to equations according to plane and antiplane strains in the x, y-plane. A slower variation of the stress-strain state with respect to z than to x and y is a distinctive feature of these. In all remaining cases the rate of change with respect to z is higher than with respect to x and y.

Variants 3 — 13 correspond to stress-strain states with the same rate of change with respect to x and y. The five variants 3 — 7 correspond to the predominance of stress-strain states symmetric about the plane $z = 0$. Of these, variant 3 corresponds to the state in which σ_z dominates in all unknown functions, variant 4 shows the predominance of u, variant 5 defines the state with equal contributions from u and σ_z, variant 6 relates to the state with simultaneous dominance of u and v, and variant 7 indicates the simultaneous dominance of u, v, σ_z.

Variant 6 reduces to the well known equations of the generalized stress-strain state. Let us consider the remaining variants.

V a r i a n t *3*. The first approximation equations are of the form

$$\frac{\partial\sigma_{x1}}{\partial x}+\frac{\partial\tau_{xy1}}{\partial y}+\frac{\partial\tau_{xz1}}{\partial z}=0,\quad 0=\sigma_{x1}-\nu(\sigma_{y1}+\sigma_{z1}) \tag{2.7}$$

$$\frac{\partial\tau_{xy1}}{\partial x}+\frac{\partial\sigma_{y1}}{\partial y}+\frac{\partial\tau_{yz1}}{\partial z}=0,\quad 0=\sigma_{y1}-\nu(\sigma_{x1}+\sigma_{z1})$$

$$\frac{\partial\sigma_{z1}}{\partial z}=0,\quad E\frac{\partial w_1}{\partial z}=\sigma_{z1}-\nu(\sigma_{x1}+\sigma_{y1})$$

$$\tau_{xy1}=0,\quad G\left(\frac{\partial u_1}{\partial z}+\frac{\partial w_1}{\partial x}\right)=\tau_{xz1},\quad G\left(\frac{\partial v_1}{\partial z}+\frac{\partial w_1}{\partial y}\right)=\tau_{yz1}$$

which define the stress-strain state of a layer whose face surfaces $z = \pm h$ are subjected to stress $\sigma_z = q$ smoothly varying (or constant) with respect to x and y.

V a r i a n t 7. First approximation equations are of the form

$$\frac{\partial\sigma_{x1}}{\partial x}+\frac{\partial\tau_{xy1}}{\partial y}+\frac{\partial\tau_{xz1}}{\partial z}=0,\quad E\frac{\partial u_1}{\partial x}=\sigma_{x1}-\nu(\sigma_{y1}+\sigma_{z1}) \tag{2.8}$$

$$\frac{\partial\tau_{xy1}}{\partial x}+\frac{\partial\sigma_{y1}}{\partial y}+\frac{\partial\tau_{yz1}}{\partial z}=0,\quad E\frac{\partial v_1}{\partial y}=\sigma_{y1}-\nu(\sigma_{x1}+\sigma_{z1})$$

$$\frac{\partial \sigma_{z1}}{\partial z} = 0, \quad E\frac{\partial w_1}{\partial z} = \sigma_{z1} - \nu(\sigma_{x1} + \sigma_{y1})$$
$$G\left(\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x}\right) = \tau_{xy1}, \quad \frac{\partial u_1}{\partial z} = 0, \quad \frac{\partial v_1}{\partial z} = 0$$

Integrating these with respect to z with boundary conditions $\sigma_z = q$, $\tau_{xz} = \tau_x$, $\tau_{yz} = \tau_y$ satisfied at $z = h$ and symmetric conditions at $z = -h$, we obtain for functions u_1 and v_1 the equations

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2 u_1}{\partial y^2} + \frac{1+\nu}{2}\frac{\partial^2 v_1}{\partial x\,\partial y} = -\frac{1-\nu^2}{E}\frac{\tau_x}{h} - \frac{\nu(1+\nu)}{E}\frac{\partial q}{\partial x} \tag{2.9}$$
$$\frac{\partial^2 v_1}{\partial y^2} + \frac{1-\nu}{2}\frac{\partial^2 v_1}{\partial x^2} + \frac{1+\nu}{2}\frac{\partial^2 u_1}{\partial x\,\partial y} = -\frac{1-\nu^2}{E}\frac{\tau_y}{h} - \frac{\nu(1+\nu)}{E}\frac{\partial q}{\partial y}$$

in which the generalized plane [tension] and compression stress states are combined. Using for surface stresses the notation $\sigma_z = q - w = W$ we obtain from (2.9) another variant of two-dimensional equations

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{1-2\nu}{2(1-\nu)}\frac{\partial^2 u_1}{\partial y^2} + \frac{1}{2(1-\nu)}\frac{\partial^2 v_1}{\partial x\,\partial y} = -\frac{(1+\nu)(1-2\nu)}{E(1-\nu)}\frac{\tau_x}{h} - \frac{\nu}{(1-\nu)h}\frac{\partial W}{\partial x} \tag{2.10}$$
$$\frac{\partial^2 v_1}{\partial y^2} + \frac{1-2\nu}{2(1-\nu)}\frac{\partial^2 v_1}{\partial x^2} + \frac{1}{2(1-\nu)}\frac{\partial^2 u_1}{\partial x\,\partial y} = -\frac{(1+\nu)(1-2\nu)}{E(1-\nu)}\frac{\tau_y}{h} - \frac{\nu}{(1-\nu)h}\frac{\partial W}{\partial y}$$

which for $\tau_x = \tau_y = 0$ and $W = 0$ become equations of plane strain, i.e. the same as the equations for u_1 and v_1 in variant 1.

Without going into the details of variants 4 and 5, we point out that the first of these yields equations of the generalized plane stress state; then, with conditions $\sigma_z = q$, $\tau_{xz} = \tau_x$, $v = V$ specified at the face $z = h$ (and symmetric conditions at $z = -h$), the equation for u_1 assumes the form

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2 u_1}{\partial y^2} = -\frac{1-\nu^2}{E}\frac{\tau_x}{h} \tag{2.11}$$

Variant 5 corresponds to the generalized simplified plane compression stress state.

Variants 8 — 12 correspond to the predominance of stress-strain states that are antisymmetric relative to the plane $z = 0$. Of these, variant 8 corresponds to the predominance of w over the remaining unknown functions, variant 9 to the predominance of τ_{xz}, variant 10 defines the state with equal contributions from w and τ_{xz}, and variants 11 and 12 to the simultaneous predominance of τ_{xz}, τ_{yz} and w, τ_{xz}, τ_{yz}, respectively.

Variant 8 was investigated in [1]; it reduces to the classic equations of plate bending. Let us consider the remaining variants.

V a r i a n t *11*. The first approximation equations are of the form

$$\frac{\partial \tau_{xz1}}{\partial z} = 0, \quad E\frac{\partial u_1}{\partial x} = \sigma_{x1} - \nu(\sigma_{y1} + \sigma_{z1}) \tag{2.12}$$

$$\frac{\partial \tau_{yz1}}{\partial z}=0,\quad E\frac{\partial v_1}{\partial y}=\sigma_{y1}-\nu(\sigma_{x1}+\sigma_{z1})$$
$$\frac{\partial \tau_{xz1}}{\partial x}+\frac{\partial \tau_{yz1}}{\partial y}+\frac{\partial \sigma_{z1}}{\partial z}=0,\quad E\frac{\partial w_1}{\partial z}=\sigma_{z1}-\nu(\sigma_{x1}+\sigma_{y1})$$
$$G\left(\frac{\partial u_1}{\partial y}+\frac{\partial v_1}{\partial x}\right)=\tau_{xy1},\quad G\frac{\partial u_1}{\partial z}=\tau_{xz1},\quad G\frac{\partial v_1}{\partial z}=\tau_{yz1}$$

that define the stress-strain state of layer $z=\pm h$ subjected to smoothly changing (or constant) tangential stresses $\tau_{xz}=\tau_x$, $\tau_{yz}=\tau_y$, i.e. to shear.

Variant *12*. The first approximation equations are of the form

$$\frac{\partial \tau_{xz1}}{\partial z}=0,\quad E\frac{\partial u_1}{\partial x}=\sigma_{x1}-\nu(\sigma_{y1}+\sigma_{z1}) \tag{2.13}$$
$$\frac{\partial \tau_{yz1}}{\partial z}=0,\quad E\frac{\partial v_1}{\partial y}=\sigma_{y1}-\nu(\sigma_{x1}+\sigma_{z1})$$
$$\frac{\partial \tau_{xz1}}{\partial x}+\frac{\partial \tau_{yz1}}{\partial y}+\frac{\partial \sigma_{z1}}{\partial z}=0,\quad \frac{\partial w_1}{\partial z}=0$$
$$G\left(\frac{\partial u_1}{\partial y}+\frac{\partial v_1}{\partial z}\right)=\tau_{xy1},\quad G\left(\frac{\partial u_1}{\partial z}+\frac{\partial w_1}{\partial x}\right)=\tau_{xz1},\quad G\left(\frac{\partial v_1}{\partial z}+\frac{\partial w_1}{\partial y}\right)=\tau_{yz1}$$

They define the stress-strain state under combined bending and shear. If only the first approximation is taken into consideration, nontrivial results are obtained when conditions $\sigma_z=q$, $u=U$, $v=V$ are specified for the plane $z=h$ (and antisymmetric conditions for $z=-h$). After integration with respect to z, we obtain for w_1 the equation

$$\frac{\partial^2 w_1}{\partial x^2}+\frac{\partial^2 w_1}{\partial y^2}=-\frac{q}{Gh}-\frac{1}{h}\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}\right) \tag{2.14}$$

When $q=0$ and $U=V=0$, we obtain from (2.14) an equation which coincides with the equation for w_1 of variant 2, i.e., we have the equation of antiplane strain.

Omitting the detailed examination of variants 9 and 10, we point out that the former corresponds to simplified shear and the latter to simplified bending with shear. The indicated antisymmetric stress-strain state in variant 9 combines with the symmetric state of the same order which corresponds to variant 7.

Variant 13 is a combination of variants 7 and 8.

The last three variants 14 – 16 correspond to stress-strain states with a higher rate of change with respect to x than to y, which is characteristic for boundary layers. It can be said that variant 14 compared with the previously described variants indicates the predominance of the cylindrical bending stress-strain state, while variant 15 corresponds to a combination of the generalized one-dimensional plane stress state and of cylindrical bending with shear. Variant 16 represents the combination of generalized one-dimensional plane compression stress state and of cylindrical bending.

Let us consider variant 14 in more detail. For it the first approximation equations are of the form

$$\frac{\partial \sigma_{x1}}{\partial x}+\frac{\partial \tau_{xz1}}{\partial z}=0,\quad E\frac{\partial u_1}{\partial x}=\sigma_{x1}-\nu\sigma_{y1} \tag{2.15}$$

$$\frac{\partial \tau_{yz1}}{\partial z} = 0, \quad 0 = \sigma_{y1} - \nu\sigma_{x1}$$

$$\frac{\partial \tau_{xz1}}{\partial x} + \frac{\partial \tau_{yz1}}{\partial y} + \frac{\partial \sigma_{z1}}{\partial z} = 0, \quad \frac{\partial w_1}{\partial z} = 0$$

$$G\left(\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x}\right) = \tau_{xy1}, \quad \frac{\partial u_1}{\partial z} + \frac{\partial w_1}{\partial x} = 0, \quad G\left(\frac{\partial v_1}{\partial z} + \frac{\partial w_1}{\partial y}\right) = \tau_{yz1}$$

whose integration with respect to z and with boundary conditions $\sigma_z = q$, $\tau_{xz} = \tau_x$, $\tau_{yz} = \tau_y$ satisfied at $z = h$ (and antisymmetric at $z = -h$) yields for w_1 the equation

$$\frac{Eh^3}{3(1-\nu^2)} \frac{\partial^4 w_1}{\partial x^4} = q + h\left(\frac{\partial \tau_x}{\partial x} + \frac{\partial \tau_y}{\partial y}\right) \tag{2.16}$$

If at $z = h$ $\tau_{yz} = \tau_y$ is specified instead of $v = V$, we have

$$\frac{Eh^3}{3(1-\nu^2)} \frac{\partial^4 w_1}{\partial x^4} - Gh\frac{\partial^2 w_1}{\partial y^2} = q + h\frac{\partial \tau_x}{\partial x} + G\frac{\partial V}{\partial y} \tag{2.17}$$

This equation defines the combination of classic bending along the x-axis and bending with shear along the y-axis.

The examples considered here show that an automated search of asymptotic integration parameters enables us to find a reasonably complete set of simplified equations that complement each other.

We note in conclusion that the problem of constructing a complete set of simplified equations cannot be formalized to the end, since much depends on the selection of the form of input equations and of suitable small parameters. However, after completion of preliminary work, the search of simplified equations can be automated using the proposed method. The combination of the substantial and formal approaches makes possible the analysis of highly complex systems of equations, with the basic volume of work carried out on a computer.

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